An Implementation of Distributive Relaxation of Stokes Equations in FEM

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ABSTRACT: We present a finite element formulation of Achi Brandt's "distributive relaxation" for the Stokes equations. The transformed system we get is almost block-lower triangular, and the principal diagonals are all Laplace-like matrices. The distributive relaxation is used as a smoother for the multigrid method. The effectiveness of this smoother is shown by numerical results.

1 Introduction

In stead of solving (smoothing) the original Stokes equations

$$\begin{pmatrix} -\Delta & \nabla \\ -\nabla\cdot & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

We make the following transformation

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} I & \nabla \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \psi \end{pmatrix}$$

Then, we are going to smooth

$$\begin{pmatrix} -\Delta & -\Delta\nabla + \nabla\Delta \\ -\nabla\cdot & \Delta \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \psi \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

Notice $-\Delta\nabla + \nabla\Delta$ is zero inside the region, then we only need to smoother two Laplace problems.

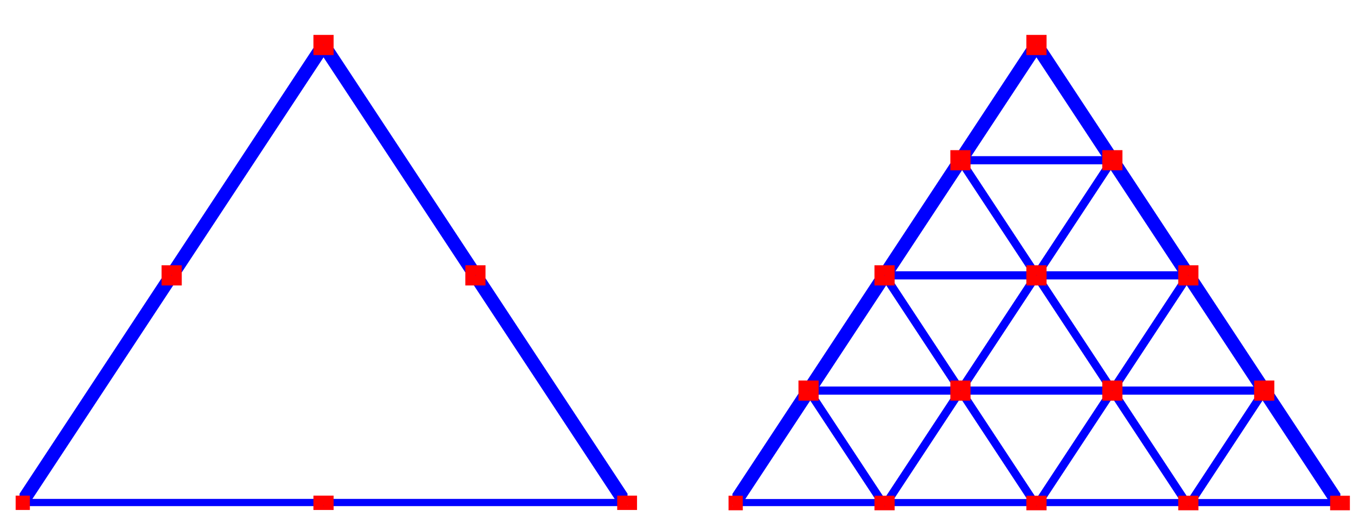
2 Finite Element Method Case

For most stable finite element pairs, the discrete $-\Delta\nabla+\nabla\Delta$ is not zero inside the region. Distributive relaxation seems not applicable. We choose the continuous piecewise quadratic functions space S_h on mesh $\mathcal{T}_h$ to approximate the pressure, and V_h be the space of continuous piecewise linear functions w.r.t. the mesh $\mathcal{T}_{h/4}$. Let $\psi_h \in S_h$, and Q_h is a quasi-interpolant to transfer the derivatives of ψ_h into the space V_h with appropriate boundary conditions.

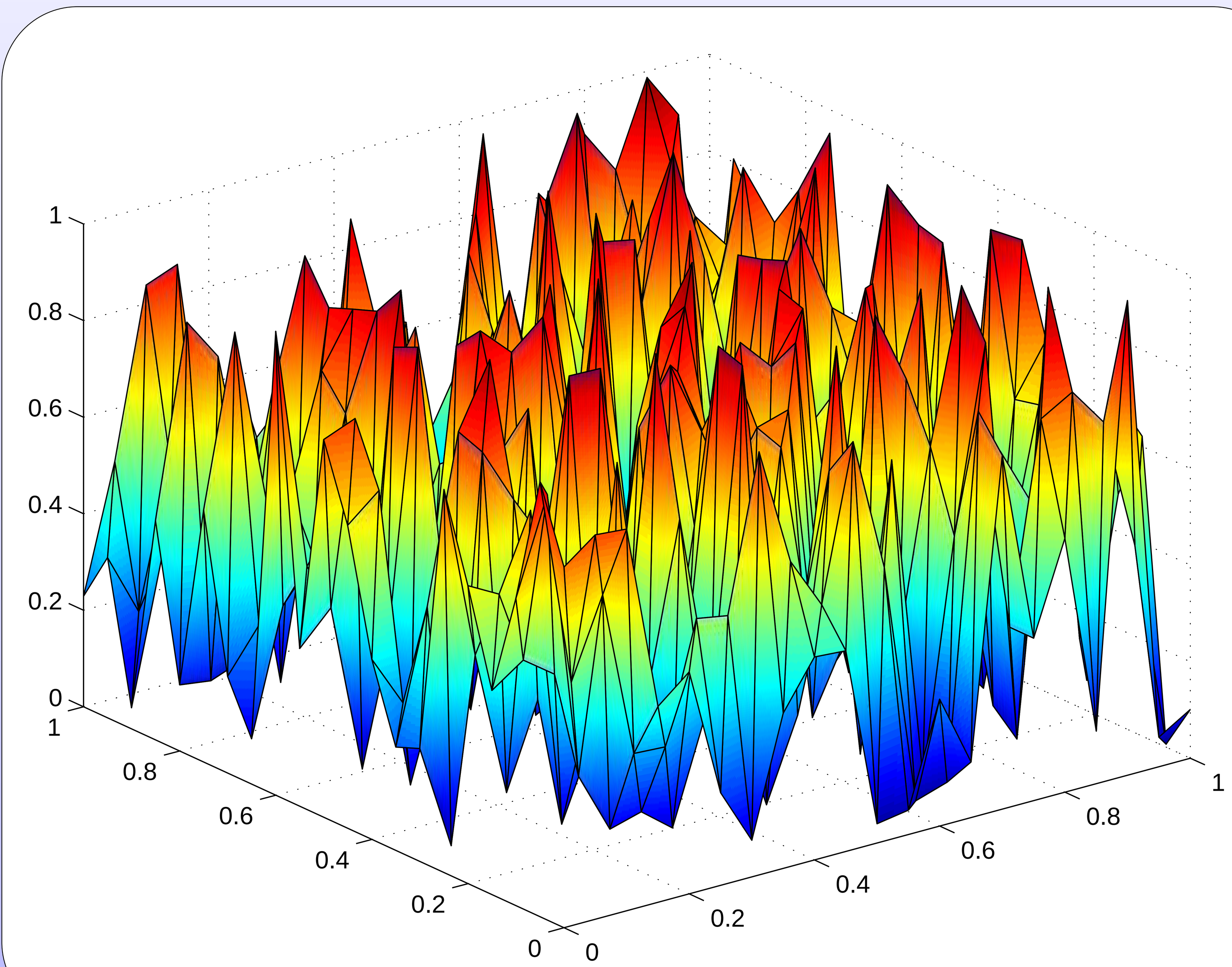
$$\mathbf{u}_h = \mathbf{w}_h + Q_h \nabla \psi_h;$$
$$G_h p_h = -C_h \psi_h;$$

G_h is the mass matrix on S_h and C_h is the discrete Laplace operator. Then the discrete case of $-\Delta\nabla + \nabla\Delta$, i.e. $|(\nabla Q_h \nabla \psi_h, \nabla\chi) + (G_h^{-1} C_h \psi_h, \nabla\chi)|, \forall\, \psi_h \in S_h, \chi \in V_h$ is bounded by ch^{α}, which is, very small.

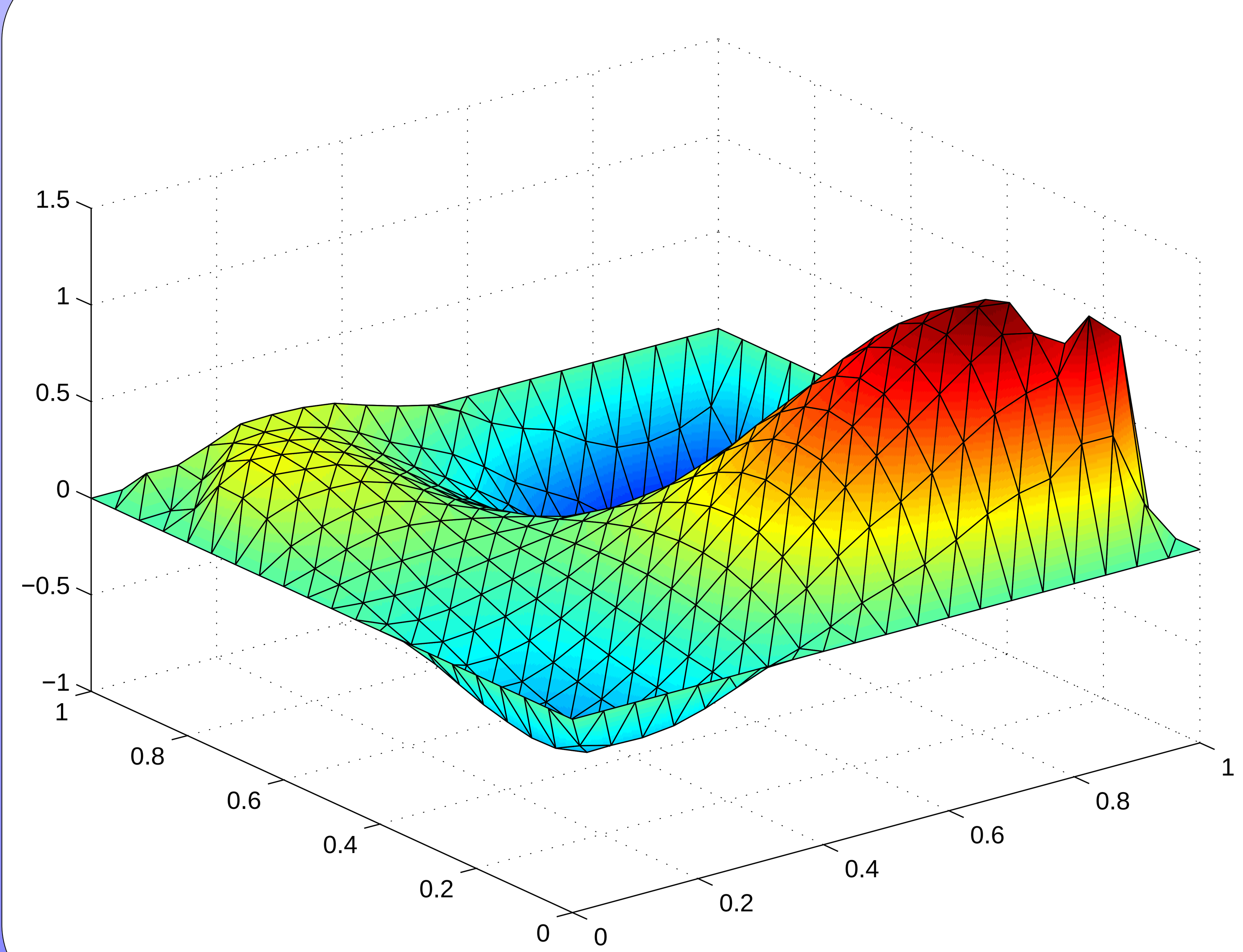
C^0 P_2 Element (Pressure) Composite C^0 P_1 Element (Velocity)



Finite Element Spaces



A random velocity before smoothing



After 5 steps of Distributive Relaxation

3 Smoothing Procedure in Matrix Form

Let T_h be the matrix of the operator $Q_h\nabla$. A is from $(\nabla\mathbf{u}, \nabla\mathbf{v})$, B is from $(\nabla p, \mathbf{v})$. We want to smooth

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

Then starting from (u_0, p_0)

- Solve (smooth) ψ_0 from the equation $T^t B \psi_0 = g - Bu_0$, and let $u_{1/2} = u_0 + T\psi_0$,
- Solve (smooth) $\widetilde{p}$ from the equation $T^t B \widetilde{p} = T^t(f - Au_{1/2} - B^t p_0)$, and let $p_1 = p_0 + \widetilde{p}$,
- Solve (smooth) $\overline{u}$ from the equation $A\overline{u} = f - Au_{1/2} - B^t p_1$, with dirichlet boundary condition $\overline{u} = -u_{1/2}$ on $\partial\Omega$. Let $u_1 = u_{1/2} + \overline{u}$.

Notice for all three steps, we are smoothing Laplace-like problems.

4 Numerical Results

A model problem is computed in $\Omega = (0,1)\times(0,1)$, with known solutions

$$\mathbf{u} = \begin{pmatrix} 2x^2(1-x)^2 y(1-y)(1-2y); \\ -2x(1-x)(1-2x)y^2(1-y)^2; \end{pmatrix} \qquad p = x^2 - y^2.$$

A two-grid method is presented, with the fine grids for pressure are $h = 1/4, h = 1/8, h = 1/16, h = 1/32$, the fine grids for velocity are $1/16, 1/32, 1/64, 1/128$ respectively. We display, the error norm of velocity and pressure with true solutions, and the ratio of these norms to their values at the end of previous cycle. Actual size errors from direct solution method is also presented to compare. We use one step pre and post distributive relaxiation as smoothing, in each step of distributive relaxation, we use one step Jacobi iteration to smooth the Laplace-like problem.

	h=1/4				h=1/8			
Iteration	ℓ^2error (u)	ratio	ℓ^2 error (p)	ratio	ℓ^2 error (u)	ratio	ℓ^2 error (p)	ratio
0	0.1244		4.2695		0.2488		7.6295	
1	0.0085	0.068	0.2049	0.479	0.0051	0.020	0.1392	0.018
2	0.0020	0.235	0.1507	0.735	7.46E-4	0.146	0.0657	0.047
3	0.0018	0.900	0.1280	0.849	6.67E-4	0.894	0.0529	0.805
4	0.0017	0.944	0.1100	0.859	6.48E-4	0.971	0.0448	0.846
Direct M	0.0013		0.032		6.00E-4		0.0173	
	h=1/16				h=1/32			
Iteration	ℓ^2 error (u)	ratio	ℓ^2 error (p)	ratio	ℓ^2 error (u)	ratio	ℓ^2 error (p)	ratio
0	0.4977		14		0.9953		27.8579	
1	0.0027	0.005	0.1018	0.007	0.0013	0.001	0.0732	0.003
2	3.16E-4	0.117	0.0334	0.328	1.45E-4	0.112	0.0182	0.248
3	2.96E-4	0.936	0.0266	0.796	1.41E-4	0.972	0.0144	0.791
4	2.94E-4	0.993	0.0228	0.807	1.41E-4	1	0.0126	0.875
Direct M	2.88E-4		0.0112		1.40E-4		0.0077	

ℓ^2 error (u) and ℓ^2 error (u) are the vector norms of errors between true solutions and computerd solutions of every step.

This work was performed under the auspices of the U.S. Department of Energy by University of California Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48

UCRL-POST-214189